PROBLEMS FOR SECTION 4.7

SM **1.** Find all integer roots of the following equations:

(a) $x^4 - x^3 - 7x^2 + x + 6 = 0$ (b) $2x^3 + 11x^2 - 7x - 6 = 0$

(c) $x^4 + x^3 + 2x^2 + x + 1 = 0$ (d) $\frac{1}{4}x^3 - \frac{1}{4}x^2 - x + 1 = 0$

2. Find all integer roots of the following equations:

(a) $x^2 + x - 2 = 0$ (b) $x^3 - x^2 - 25x + 25 = 0$ (c) $x^5 - 4x^3 - 3 = 0$

SM **3.** Perform the following divisions:

(a) $(2x^3 + 2x - 1) \div (x - 1)$ (b) $(x^4 + x^3 + x^2 + x) \div (x^2 + x)$

(c) $(x^5 - 3x^4 + 1) \div (x^2 + x + 1)$ (d) $(3x^8 + x^2 + 1) \div (x^3 - 2x + 1)$

SM **4.** Find possible formulas for each of the three polynomials with graphs shown in Fig. 6.

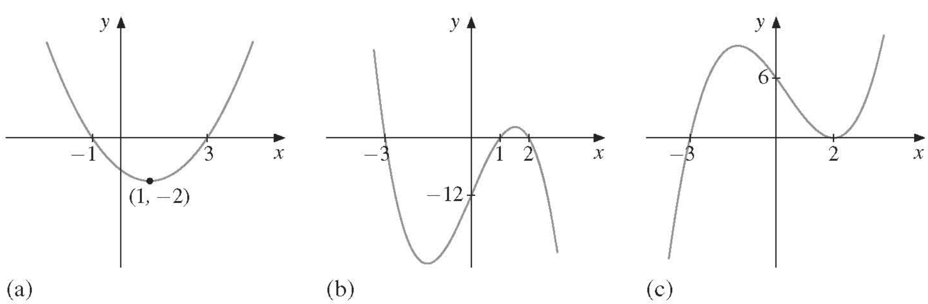


Figure 6

5. Perform the following divisions:

(a) $(x^2 - x - 20) \div (x - 5)$ (b) $(x^3 - 1) \div (x - 1)$ (c) $(-3x^3 + 48x) \div (x - 4)$

6. Show that the division $(x^4 + 3x^2 + 5) \div (x - c)$ leaves a remainder for all values of c.

7. Prove that $R(x) = \dfrac{ax + b}{cx + d} = \dfrac{a}{c} + \dfrac{bc - ad}{c(cx + d)}$ $(c \neq 0)$.

SM **8.** The following function has been used in demand theory:

$$E = \alpha \frac{x^2 - \gamma x}{x + \beta} \qquad (\alpha, \beta, \text{ and } \gamma \text{ are constants})$$

Perform the division $(x^2 - \gamma x) \div (x + \beta)$, and use the result to express E as a sum of a linear function and a proper fraction.

the half of the parabola $y = x^2$ shown in Fig. 4.3.6.) Finally, for $r < 0$, the graph is shown in Fig. 3. (For $r = -1$, the graph is half of the hyperbola $y = 1/x$ shown in Fig. 4.3.9.)

Figure 4 illustrates how the graph of $y = x^r$ changes with changing positive values of the exponent.

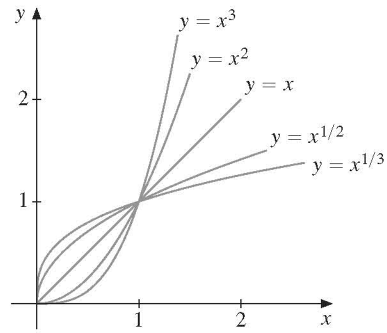


Figure 4

PROBLEMS FOR SECTION 4.8

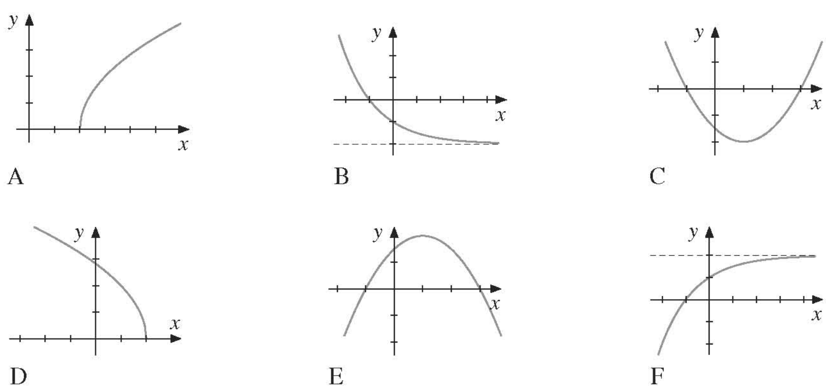
1. Sketch the graphs of $y = x^{-3}$, $y = x^{-1}$, $y = x^{-1/2}$, and $y = x^{-1/3}$, defined for $x > 0$.

2. Use a calculator to find approximate values for (a) $\sqrt{2}^{\sqrt{2}}$ (b) π^{π}

3. Solve the following equations for x: (a) $2^{2x} = 8$ (b) $3^{3x+1} = 1/81$ (c) $10^{x^2-2x+2} = 100$

(SM) **4.** Match each of the graphs A–F with one of the functions (a)–(f) in the following table. (In (f) try to find a suitable function which has the remaining graph.)

(a) $y = \frac{1}{2}x^2 - x - \frac{3}{2}$ has graph ___	(b) $y = 2\sqrt{2 - x}$ has graph ___
(c) $y = -\frac{1}{2}x^2 + x + \frac{3}{2}$ has graph ___	(d) $y = \left(\frac{1}{2}\right)^x - 2$ has graph ___
(e) $y = 2\sqrt{x - 2}$ has graph ___	(f) $y =$ has graph ___



5. Find t when (a) $3^{5t}9^t = 27$ (b) $9^t = (27)^{1/5}/3$

PROBLEMS FOR SECTION 4.9

1. If the population of Europe grew at the rate of 0.72% annually, what would be the doubling time?

2. The population of Botswana was estimated to be 1.22 million in 1989, and to be growing at the rate of 3.4% annually. If $t = 0$ denotes 1989, find a formula for the population $P(t)$ at date t. What is the doubling time?

3. A savings account with an initial deposit of \$100 earns 12% interest per year. What is the amount of savings after t years? Make a table similar to Table 1. (Stop at 50 years.)

4. Fill in the following table and sketch the graphs of $y = 2^x$ and $y = 2^{-x}$.

x	-3	-2	-1	0	1	2	3
2^x							
2^{-x}							

5. Use your calculator to fill in the following table:

x	-2	-1	0	1	2
$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$					

Use it to find five points on the "bell curve" graph of

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad \text{(the \textbf{normal density function})}$$

which is one of the most important functions in statistics.

6. The area of Zimbabwe is approximately $3.91 \cdot 10^{11}$ square metres. Referring to the text at the end of Example 1 and using a calculator, solve the equation $5.1 \cdot 10^6 \cdot 1.035^t = 3.91 \cdot 10^{11}$ for t, and interpret your answer. (Recall that $t = 0$ corresponds to 1969.)

7. With $f(t) = Aa^t$, if $f(t + t^*) = 2f(t)$, prove that $a^{t^*} = 2$. (This shows that the doubling time t^* of the general exponential function is independent of the initial time t.)

8. Which of the following equations do *not* define exponential functions of x?

(a) $y = 3^x$ (b) $y = x^{\sqrt{2}}$ (c) $y = (\sqrt{2})^x$ (d) $y = x^x$ (e) $y = (2.7)^x$ (f) $y = 1/2^x$

9. Suppose that all prices rise at the same proportional rate in a country whose inflation rate is 19% per year. For an item that currently costs P_0, use the implied formula $P(t) = P_0(1.19)^t$ for the price after t years in order to predict the prices of:

(a) A 20 kg bag of corn, presently costing \$16, after 5 years.

(b) A \$4.40 can of coffee after 10 years. (c) A \$250 000 house after 4 years.

10. Find possible exponential functions for the graphs in Fig. 4.

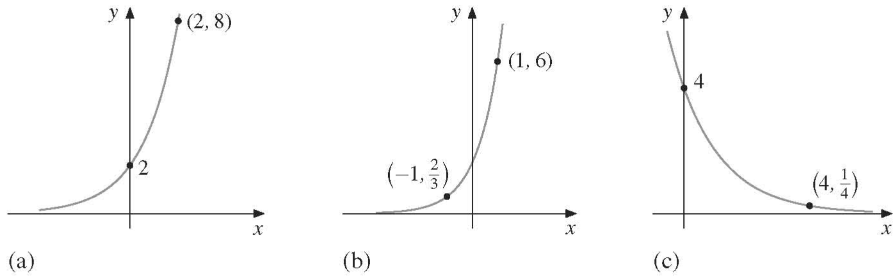


Figure 4

4.10 Logarithmic Functions

The doubling time of an exponential function $f(t) = Aa^t$ was defined as the time it takes for $f(t)$ to become twice as large. In order to find the doubling time t^*, we must solve the equation $a^{t^*} = 2$ for t^*. In economics, we often need to solve similar problems:

A. At the present rate of inflation, how long will it take the price level to triple?

B. If the world's population grows at 2% per year, how long does it take to double its size?

C. If \$1000 is invested in a savings account bearing interest at the annual rate of 8%, how long does it take for the account to reach \$10 000?

All these questions involve solving equations of the form $a^x = b$ for x. For instance, problem C reduces to the problem of finding which x solves the equation $1000(1.08)^x = 10\,000$.

We begin with equations in which the base of the exponential is e, which was, as you recall, the irrational number 2.718 .... Here are some examples:

$$\text{(i)}\quad e^x = 4 \qquad \text{(ii)}\quad 5e^{-3x} = 16 \qquad \text{(iii)}\quad A\alpha e^{-\alpha x} = k$$

In all these equations, the unknown x occurs as an exponent. We therefore introduce the following useful definition. If $e^u = a$, we call u the **natural logarithm** of a, and we write $u = \ln a$. Hence, we have the following definition of the symbol $\ln a$:

$$e^{\ln a} = a \qquad (a \text{ is any positive number}) \tag{1}$$

Thus, $\ln a$ is the power of e you need to get a. In particular, if $e^x = 4$, then x must be $\ln 4$.

Because e^u is a strictly increasing function of u, it follows that $\ln a$ *is* uniquely determined by the definition (1). You should memorize this definition. It is the foundation for everything until the use of mechanical and then electronic calculators became widespread, tables of logarithms to the base 10 were frequently used to simplify complicated calculations involving multiplication, division, square roots, and so on.

Suppose that a is a fixed positive number (usually chosen > 1). If $a^u = x$, then we call u the **logarithm of x to base a** and write $u = \log_a x$. The symbol $\log_a x$ is then defined for every positive number x by the following:

$$a^{\log_a x} = x \tag{4}$$

For instance, $\log_2 32 = 5$, because $2^5 = 32$, whereas $\log_{10}(1/100) = -2$, because $10^{-2} = 1/100$. Note that $\ln x$ is $\log_e x$.

By taking the ln on each side of (4), we obtain $\log_a x \cdot \ln a = \ln x$, so that

$$\log_a x = \frac{1}{\ln a} \ln x \tag{5}$$

This reveals that the logarithm of x in the system with base a is proportional to $\ln x$, *with a proportionality factor* $1/\ln a$. It follows immediately that $\log_a$ obeys the same rules as ln:

$$\begin{array}{ll} \text{(a)}\ \log_a(xy) = \log_a x + \log_a y, & \text{(b)}\ \log_a(x/y) = \log_a x - \log_a y \\ \text{(c)}\ \log_a x^p = p \log_a x, & \text{(d)}\ \log_a 1 = 0 \quad \text{and} \quad \log_a a = 1 \end{array} \tag{6}$$

For example, (a) follows from the corresponding rule for ln:

$$\log_a(xy) = \frac{1}{\ln a}\ln(xy) = \frac{1}{\ln a}(\ln x + \ln y) = \frac{1}{\ln a}\ln x + \frac{1}{\ln a}\ln y = \log_a x + \log_a y$$

PROBLEMS FOR SECTION 4.10

1. Express as multiples of ln 3: (a) $\ln 9$ (b) $\ln \sqrt{3}$ (c) $\ln \sqrt[5]{3^2}$ (d) $\ln \dfrac{1}{81}$

2. Solve the following equations for x:

   (a) $3^x = 8$ (b) $\ln x = 3$ (c) $\ln(x^2 - 4x + 5) = 0$

   (d) $\ln[x(x-2)] = 0$ (e) $\dfrac{x\ln(x+3)}{x^2+1} = 0$ (f) $\ln(\sqrt{x} - 5) = 0$

SM 3. Solve the following equations for x:

   (a) $3^x 4^{x+2} = 8$ (b) $3\ln x + 2\ln x^2 = 6$ (c) $4^x - 4^{x-1} = 3^{x+1} - 3^x$

   (d) $\log_2 x = 2$ (e) $\log_x e^2 = 2$ (f) $\log_3 x = -3$

SM 4. (a) Let $f(t) = Ae^{rt}$ and $g(t) = Be^{st}$, with $A > 0$, $B > 0$, and $r \neq s$. Solve the equation $f(t) = g(t)$ for t.

   (b) In 1990 the GNP (gross national product) of China was estimated to be $1.2 \cdot 10^{12}$ US dollars, and the rate of growth was estimated to be $r = 0.09$. By comparison, the GNP for the USA was reported as $\$5.6 \cdot 10^{12}$, with an estimated rate of growth of $s = 0.02$. If the GNP of each country continued to grow exponentially at the rates $r = 0.09$ and $s = 0.02$, respectively, when would the GNP of the two nations be the same?

5. Which of the following formulas are always true and which are sometimes false (all variables are positive)?

   (a) $(\ln A)^4 = 4\ln A$ (b) $\ln B = 2\ln\sqrt{B}$ (c) $\ln A^{10} - \ln A^4 = 3\ln A^2$

6. Which of the following formulas are always true and which are sometimes false (all variables are positive)?

(a) $\ln \dfrac{A+B}{C} = \ln A + \ln B - \ln C$ (b) $\ln \dfrac{A+B}{C} = \ln(A+B) - \ln C$

(c) $\ln \dfrac{A}{B} + \ln \dfrac{B}{A} = 0$ (d) $p \ln(\ln A) = \ln(\ln A^p)$

(e) $p \ln(\ln A) = \ln(\ln A)^p$ (f) $\dfrac{\ln A}{\ln B + \ln C} = \ln A(BC)^{-1}$

7. Simplify the following expressions:

(a) $\exp[\ln(x)] - \ln[\exp(x)]$ (b) $\ln[x^4 \exp(-x)]$ (c) $\exp[\ln(x^2) - 2\ln y]$

REVIEW PROBLEMS FOR CHAPTER 4

1. (a) Let $f(x) = 3 - 27x^3$. Compute $f(0),\ f(-1),\ f(1/3)$, and $f(\sqrt[3]{2})$.

(b) Show that $f(x) + f(-x) = 6$ for all x.

2. (a) Let $F(x) = 1 + \dfrac{4x}{x^2+4}$. Compute $F(0),\ F(-2),\ F(2)$, and $F(3)$.

(b) What happens to $F(x)$ when x becomes large positive or negative?

(c) Give a rough sketch of the graph of F.

3. Figure A combines the graphs of a quadratic function f and a linear function g. Use the graphs to find those x where: (i) $f(x) \le g(x)$ (ii) $f(x) \le 0$ (iii) $g(x) \ge 0$.

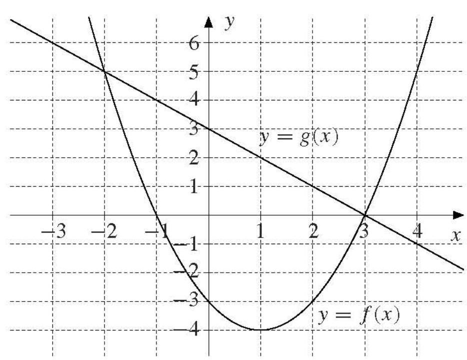


Figure A

4. Find the domains of:

(a) $f(x) = \sqrt{x^2 - 1}$ (b) $g(x) = \dfrac{1}{\sqrt{x-4}}$ (c) $h(x) = \sqrt{(x-3)(5-x)}$

5. (a) The cost of producing x units of a commodity is given by $C(x) = 100 + 40x + 2x^2$. Find $C(0),\ C(100)$, and $C(101) - C(100)$.

(b) Find $C(x+1) - C(x)$, and explain in words the meaning of the difference.

6. Find the slopes of the straight lines (a) $y = -4x + 8$ (b) $3x + 4y = 12$ (c) $\frac{x}{a} + \frac{y}{b} = 1$

7. Find equations for the following straight lines:

(a) L_1 passes through $(-2, 3)$ and has a slope of -3.

(b) L_2 passes through $(-3, 5)$ and $(2, 7)$.

(c) L_3 passes through (a, b) and $(2a, 3b)$ (suppose $a \neq 0$).

8. If $f(x) = ax + b$, $f(2) = 3$, and $f(-1) = -3$, then $f(-3) = ?$

9. Fill in the following table, then make a rough sketch of the graph of $y = x^2 e^x$.

x	-5	-4	-3	-2	-1	0	1
$y = x^2 e^x$							

10. Find the equation for the parabola $y = ax^2 + bx + c$ that passes through the three points $(1, -3)$, $(0, -6)$, and $(3, 15)$. (*Hint:* Determine a, b, and c.)

11. (a) If a firm sells Q tons of a product, the price P received per ton is $P = 1000 - \frac{1}{3}Q$. The price it has to pay per ton is $P = 800 + \frac{1}{5}Q$. In addition, it has transportation costs of 100 per ton. Express the firm's profit π as a function of Q, the number of tons sold, and find the profit-maximizing quantity.

(b) Suppose the government imposes a tax on the firm's product of 10 per ton. Find the new expression for the firm's profits $\hat{\pi}$ and the new profit-maximizing quantity.

12. In Example 4.6.1, suppose a tax of t per unit produced is imposed. If $t < 100$, what production level now maximizes profits?

13. (a) A firm produces a commodity and receives \$100 for each unit sold. The cost of producing and selling x units is $20x + 0.25x^2$ dollars. Find the production level that maximizes profits.

(b) A tax of \$10 per unit is imposed. What is now the optimal production level?

(c) Answer the question in (b) if the sales price per unit is p, the total cost of producing and selling x units is $\alpha x + \beta x^2$, and the tax per unit is t.

(SM) **14.** Write the following polynomials as products of linear factors:

(a) $p(x) = x^3 + x^2 - 12x$ (b) $q(x) = 2x^3 + 3x^2 - 18x + 8$

15. Which of the following divisions leave no remainder? (a and b are constants; n is a natural number.)

(a) $(x^3 - x - 1)/(x - 1)$ (b) $(2x^3 - x - 1)/(x - 1)$

(c) $(x^3 - ax^2 + bx - ab)/(x - a)$ (d) $(x^{2n} - 1)/(x + 1)$

16. Find the values of k that make the polynomial $q(x)$ divide the polynomial $p(x)$:

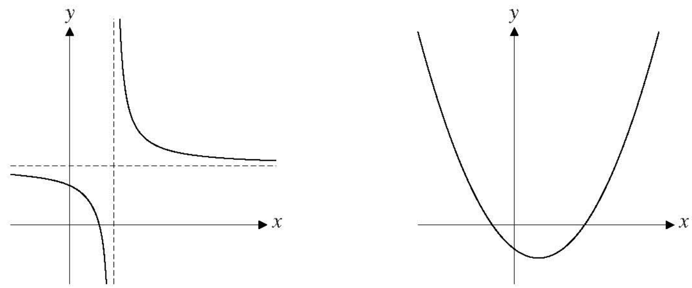
(a) $p(x) = x^2 - kx + 4$; $q(x) = x - 2$ (b) $p(x) = k^2x^2 - kx - 6$; $q(x) = x + 2$

(c) $p(x) = x^3 - 4x^2 + x + k$; $q(x) = x + 2$ (d) $p(x) = k^2x^4 - 3kx^2 - 4$; $q(x) = x - 1$

SM **17.** The cubic function $p(x) = \frac{1}{4}x^3 - x^2 - \frac{11}{4}x + \frac{15}{2}$ has three real zeros. Verify that $x = 2$ is one of them, and find the other two.

18. In 1964 a five-year plan was introduced in Tanzania. One objective was to double the real per capita income over the next 15 years. What is the average annual rate of growth of real income per capita required to achieve this objective?

SM **19.** Figure B shows the graphs of two functions f and g. Check which of the constants a, b, c, p, q, and r are > 0, $= 0$, or < 0.



$$y = f(x) = \frac{ax + b}{x + c} \qquad\qquad y = g(x) = px^2 + qx + r$$

Figure B

20. (a) Determine the relationship between the Celsius (C) and Fahrenheit (F) temperature scales when you know that (i) the relation is linear; (ii) water freezes at 0°C and 32°F; and (iii) water boils at 100°C and 212°F.

(b) Which temperature is represented by the same number in both scales?

21. Solve for t: (a) $x = e^{at+b}$ (b) $e^{-at} = 1/2$ (c) $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} = \frac{1}{8}$

SM **22.** Prove the following equalities (with appropriate restrictions on the variables):

(a) $\ln x - 2 = \ln(x/e^2)$ (b) $\ln x - \ln y + \ln z = \ln \frac{xz}{y}$

(c) $3 + 2 \ln x = \ln(e^3 x^2)$ (d) $\frac{1}{2} \ln x - \frac{3}{2} \ln \frac{1}{x} - \ln(x + 1) = \ln \frac{x^2}{x + 1}$