PROBLEMS FOR SECTION 1.6

1. Decide which of the following inequalities are true:

(a) $-6.15 > -7.16$ (b) $6 \geq 6$ (c) $(-5)^2 \leq 0$ (d) $-\frac{1}{2}\pi < -\frac{1}{3}\pi$

(e) $\frac{4}{5} > \frac{6}{7}$ (f) $2^3 < 3^2$ (g) $2^{-3} < 3^{-2}$ (h) $\frac{1}{2} - \frac{2}{3} < \frac{1}{4} - \frac{1}{3}$

2. Find what values of x satisfy:

(a) $-x - 3 \leq 5$ (b) $3x + 5 < x - 13$ (c) $3x - (x - 1) \geq x - (1 - x)$

(d) $\frac{2x - 4}{3} \leq 7$ (e) $\frac{1}{3}(1 - x) \geq 2(x - 3)$ (f) $\frac{t}{24} - (t + 1) + \frac{3t}{8} < \frac{5}{12}(t + 1)$

In Problems 3–6, solve the inequalities.

3. (a) $\frac{x + 2}{x - 1} < 0$ (b) $\frac{2x + 1}{x - 3} > 1$ (c) $5a^2 \leq 125$

SM 4. (a) $2 < \frac{3x + 1}{2x + 4}$ (b) $\frac{120}{n} + 1.1 \leq 1.85$ (c) $g^2 - 2g \leq 0$

(d) $\frac{1}{p - 2} + \frac{3}{p^2 - 4p + 4} \geq 0$ (e) $\frac{-n - 2}{n + 4} > 2$ (f) $x^4 < x^2$

SM 5. (a) $(x - 1)(x + 4) > 0$ (b) $(x - 1)^2(x + 4) > 0$ (c) $(x - 1)^3(x - 2) \leq 0$

(d) $(5x - 1)^{10}(x - 1) < 0$ (e) $(5x - 1)^{11}(x - 1) < 0$ (f) $\frac{3x - 1}{x} > x + 3$

(g) $\frac{x - 3}{x + 3} < 2x - 1$ (h) $x^2 - 4x + 4 > 0$ (i) $x^3 + 2x^2 + x \leq 0$

6. (a) $1 \leq \frac{1}{3}(2x - 1) + \frac{8}{3}(1 - x) < 16$ (b) $-5 < \frac{1}{x} < 0$ (c) $\frac{(1/x) - 1}{(1/x) + 1} \geq 1$

7. Decide whether the following inequalities are valid for all x and y:

(a) $x + 1 > x$ (b) $x^2 > x$ (c) $x + x > x$ (d) $x^2 + y^2 \geq 2xy$

8. (a) The temperature for storing potatoes should be between 4°C and 6°C. What are the corresponding temperatures in degrees Fahrenheit? (See Example 4.)

(b) The freshness of a bottle of milk is guaranteed for 7 days if it is kept at a temperature between 36°F and 40°F. Find the corresponding temperature variation in degrees Celsius.

HARDER PROBLEM

9. If a and b are two positive numbers, the numbers m_A, m_G, and m_H defined by

$$m_A = \frac{1}{2}(a + b), \qquad m_G = \sqrt{ab}, \qquad \text{and} \qquad \frac{1}{m_H} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$$

7. Give economic interpretations to each of the following expressions and then use a calculator to find the approximate values:

(a) $100 \cdot (1.01)^8$ (b) $50\,000 \cdot (1.15)^{10}$ (c) $6000 \cdot (1.03)^{-8}$

8. (a) \$100 000 is deposited into an account earning 8% interest per year. What is the amount after 10 years?

(b) If the interest rate is 8% each year, how much money should you have deposited in a bank 6 years ago to have \$25 000 today?

SM **9.** Expand and simplify:

(a) $a(a-1)$ (b) $(x-3)(x+7)$ (c) $-\sqrt{3}\left(\sqrt{3}-\sqrt{6}\right)$ (d) $\left(1-\sqrt{2}\right)^2$

(e) $(x-1)^3$ (f) $(1-b^2)(1+b^2)$ (g) $(1+x+x^2+x^3)(1-x)$ (h) $(1+x)^4$

10. Complete the following:

(a) $x^{-1}y^{-1} = 3$ implies $x^3y^3 = \cdots$ (b) $x^7 = 2$ implies $(x^{-3})^6(x^2)^2 = \cdots$

(c) $\left(\dfrac{xy}{z}\right)^{-2} = 3$ implies $\left(\dfrac{z}{xy}\right)^6 = \cdots$ (d) $a^{-1}b^{-1}c^{-1} = 1/4$ implies $(abc)^4 = \cdots$

11. Factor the expressions

(a) $25x-5$ (b) $3x^2 - x^3y$ (c) $50 - x^2$ (d) $a^3 - 4a^2b + 4ab^2$

SM **12.** Factor the expressions

(a) $5(x+2y) + a(x+2y)$ (b) $(a+b)c - d(a+b)$ (c) $ax + ay + 2x + 2y$

(d) $2x^2 - 5yz + 10xz - xy$ (e) $p^2 - q^2 + p - q$ (f) $u^3 + v^3 - u^2v - v^2u$

13. Compute the following without using a calculator:

(a) $16^{1/4}$ (b) $243^{-1/5}$ (c) $5^{1/7} \cdot 5^{6/7}$ (d) $(4^8)^{-3/16}$

(e) $64^{1/3} + \sqrt[3]{125}$ (f) $(-8/27)^{2/3}$ (g) $(-1/8)^{-2/3} + (1/27)^{-2/3}$ (h) $\dfrac{1000^{-2/3}}{\sqrt[3]{5^{-3}}}$

14. Solve the following equations for x:

(a) $2^{2x} = 8$ (b) $3^{3x+1} = 1/81$ (c) $10^{x^2-2x+2} = 100$

15. Find the unknown x in each of the following equations:

(a) $25^5 \cdot 25^x = 25^3$ (b) $3^x - 3^{x-2} = 24$ (c) $3^x \cdot 3^{x-1} = 81$

(d) $3^5 + 3^5 + 3^5 = 3^x$ (e) $4^{-6} + 4^{-6} + 4^{-6} + 4^{-6} = 4^x$ (f) $\dfrac{2^{26} - 2^{23}}{2^{26} + 2^{23}} = \dfrac{x}{9}$

SM **16.** Simplify: (a) $\dfrac{s}{2s-1} - \dfrac{s}{2s+1}$ (b) $\dfrac{x}{3-x} - \dfrac{1-x}{x+3} - \dfrac{24}{x^2-9}$ (c) $\dfrac{\dfrac{1}{x^2y} - \dfrac{1}{xy^2}}{\dfrac{1}{x^2} - \dfrac{1}{y^2}}$

SM **17.** Reduce the following fractions:

(a) $\dfrac{25a^3b^2}{125ab}$ (b) $\dfrac{x^2-y^2}{x+y}$ (c) $\dfrac{4a^2-12ab+9b^2}{4a^2-9b^2}$ (d) $\dfrac{4x-x^3}{4-4x+x^2}$

18. Solve the following inequalities:

(a) $2(x-4)<5$ (b) $\dfrac{1}{3}(y-3)+4\geq 2$ (c) $8-0.2x\leq \dfrac{4-0.1x}{0.5}$

(d) $\dfrac{x-1}{-3}>\dfrac{-3x+8}{-5}$ (e) $|5-3x|\leq 8$ (f) $|x^2-4|\leq 2$

19. Using a mobile phone costs \$30 per month, and an additional \$0.16 per minute of use.

(a) What is the cost for one month if the phone is used for a total of x minutes?

(b) What are the smallest and largest numbers of *hours* you can use the phone in a month if the monthly telephone bill is to be between \$102 and \$126?

20. If a rope could be wrapped around the Earth's surface at the equator, it would be approximately circular and about 40 million metres long. Suppose we wanted to extend the rope to make it 1 metre above the equator at every point. How many more metres of rope would be needed? (The circumference of a circle with radius r is $2\pi r$.)

21. (a) Prove that $a+\dfrac{a\cdot p}{100}-\dfrac{\left(a+\dfrac{a\cdot p}{100}\right)\cdot p}{100}=a\left[1-\left(\dfrac{p}{100}\right)^2\right]$.

(b) An item initially costs \$2000 and then its price is increased by 5%. Afterwards the price is lowered by 5%. What is the final price?

(c) An item initially costs a dollars and then its price is increased by $p\%$. Afterwards the (new) price is lowered by $p\%$. What is the final price of the item? (After considering this problem, look at the expression in part (a).)

(d) What is the result if one first *lowers* a price by $p\%$ and then *increases* it by $p\%$?

22. (a) If $a>b$, is it necessarily true that $a^2>b^2$?

(b) Show that if $a+b>0$, then $a>b$ implies $a^2>b^2$.

23. (a) If $a>b$, use numerical examples to check whether $1/a>1/b$, or $1/a<1/b$.

(b) Prove that if $a>b$ and $ab>0$, then $1/b>1/a$.

24. Prove that (i) $|ab|=|a|\cdot|b|$ and (ii) $|a+b|\leq|a|+|b|$, for all real numbers a and b. (The inequality in (ii) is called the **triangle inequality**.)

SM **25.** Consider an equilateral triangle, and let P be an arbitrary point within the triangle. Let h_1, h_2, and h_3 be the shortest distances from P to each of the three sides. Show that the sum $h_1+h_2+h_3$ is independent of where point P is placed in the triangle. (*Hint:* Compute the area of the triangle as the sum of three triangles.)

Often, solving a problem in economic analysis requires formulating an appropriate algebraic equation.

EXAMPLE 4 A firm manufactures a commodity that costs \$20 per unit to produce. In addition, the firm has fixed costs of \$2000. Each unit is sold for \$75. How many units must be sold if the firm is to meet a profit target of \$14 500?

Solution: If the number of units produced and sold is denoted by Q, then the revenue of the firm is $75Q$ and the total cost of production is $20Q + 2000$. Because profit is the difference between total revenue and total cost, it can be written as $75Q - (20Q + 2000)$. Because the profit target is 14 500, the equation

$$75Q - (20Q + 2000) = 14\,500$$

must be satisfied. It is easy to find the solution $Q = 16\,500/55 = 300$ units. ▮

PROBLEMS FOR SECTION 2.1

In Problems 1–3, solve each of the equations.

1. (a) $5x - 10 = 15$ (b) $2x - (5 + x) = 16 - (3x + 9)$
   (c) $-5(3x - 2) = 16(1 - x)$ (d) $4x + 2(x - 4) - 3 = 2(3x - 5) - 1$
   (e) $\frac{2}{3}x = -8$ (f) $(8x - 7)5 - 3(6x - 4) + 5x^2 = x(5x - 1)$
   (g) $x^2 + 10x + 25 = 0$ (h) $(3x - 1)^2 + (4x + 1)^2 = (5x - 1)(5x + 1) + 1$

2. (a) $3x + 2 = 11$ (b) $-3x = 21$ (c) $3x = \frac{1}{4}x - 7$
   (d) $\dfrac{x - 3}{4} + 2 = 3x$ (e) $\dfrac{1}{2x + 1} = \dfrac{1}{x + 2}$ (f) $\sqrt{2x + 14} = 16$

(SM) 3. (a) $\dfrac{x - 3}{x + 3} = \dfrac{x - 4}{x + 4}$ (b) $\dfrac{3}{x - 3} - \dfrac{2}{x + 3} = \dfrac{9}{x^2 - 9}$ (c) $\dfrac{6x}{5} - \dfrac{5}{x} = \dfrac{2x - 3}{3} + \dfrac{8x}{15}$

4. Solve the following problems by first formulating an equation in each case:
   (a) The sum of twice a number and 5 is equal to the difference between the number and 3. Find the number.
   (b) The sum of three successive natural numbers is 10 more than twice the smallest of them. Find the numbers.
   (c) Jane receives double pay for every hour she works over and above 38 hours per week. Last week, she worked 48 hours and earned a total of \$812. What is Jane's regular hourly wage?
   (d) James has invested \$15 000 at an annual interest rate of 10%. How much additional money should he invest at the interest rate of 12% if he wants the total interest earned by the end of the year to equal \$2100?
   (e) When Mr. Barnes passed away, 2/3 of his estate was left to his wife, 1/4 was shared by his children, and the remainder, \$100 000, was donated to a charity. How big was Mr. Barnes's estate?

Solution:

(a) It follows easily from the given equation that $(r-\gamma)^{-\delta} = (M-\alpha Y)/\beta$. Then raising each side to the power $-1/\delta$ yields

$$r-\gamma = \left(\frac{M-\alpha Y}{\beta}\right)^{-1/\delta}, \quad \text{or} \quad r = \gamma + \left(\frac{\beta}{M-\alpha Y}\right)^{1/\delta} \tag{$*$}$$

where we used the fact that $(a/b)^{-p} = (b/a)^p$.

(b) In this case $1/\delta = 1/0.84 = 100/84 = 25/21$, and the required formula follows immediately from $(*)$.

PROBLEMS FOR SECTION 2.2

1. Find the value of Y in the models (iii) and (iv) in Example 1. Verify that formula $(**)$ gives the same result.

(SM) 2. Solve the following equations for x:

(a) $\dfrac{1}{ax} + \dfrac{1}{bx} = 2$ (b) $\dfrac{ax+b}{cx+d} = A$ (c) $\dfrac{1}{2}px^{-1/2} - w = 0$

(d) $\sqrt{1+x} + \dfrac{ax}{\sqrt{1+x}} = 0$ (e) $a^2x^2 - b^2 = 0$ (f) $(3+a^2)^x = 1$

3. Solve each equation for the variable suggested:

(a) $q = 0.15p + 0.14$ for p (supply of rice in India)

(b) $S = \alpha + \beta P$ for P (supply function)

(c) $A = \frac{1}{2}gh$ for g (the area of a triangle)

(d) $V = \frac{4}{3}\pi r^3$ for r (the volume of a ball)

(e) $AK^\alpha L^\beta = Y_0$ for L (production function)

(SM) 4. Solve the following equations for the indicated variables:

(a) $\alpha x - a = \beta x - b$ for x (b) $\sqrt{pq} - 3q = 5$ for p

(c) $Y = 94 + 0.2(Y - (20 + 0.5Y))$ for Y (d) $K^{1/2}\left(\dfrac{1}{2}\dfrac{r}{w}K\right)^{1/4} = Q$ for K

(e) $\dfrac{\frac{1}{2}K^{-1/2}L^{1/4}}{\frac{1}{4}L^{-3/4}K^{1/2}} = \dfrac{r}{w}$ for L (f) $\dfrac{1}{2}pK^{-1/4}\left(\dfrac{1}{2}\dfrac{r}{w}\right)^{1/4} = r$ for K

(SM) 5. Solve for the indicated variable:

(a) $\dfrac{1}{s} + \dfrac{1}{T} = \dfrac{1}{t}$ for s (b) $\sqrt{KLM} - \alpha L = B$ for M

(c) $\dfrac{x - 2y + xz}{x - z} = 4y$ for z (d) $V = C\left(1 - \dfrac{T}{N}\right)$ for T

Consider again equation (2) with solutions x_1 and x_2 given by (3). Expanding the right-hand side of the identity $x^2 + px + q = (x - x_1)(x - x_2)$ corresponding to (5) yields $x^2 + px + q = x^2 - (x_1 + x_2)x + x_1 x_2$. Equating like powers of x gives $x_1 + x_2 = -p$ and $x_1 x_2 = q$. (The same formulas are obtained by adding and multiplying the two solutions found in (3).) Thus:

If x_1 and x_2 are the roots of $x^2 + px + q = 0$, then

$$x_1 + x_2 = -p \qquad \text{and} \qquad x_1 x_2 = q \tag{6}$$

In words, the sum of the roots is minus the coefficient of the first-order term and the product is the constant term.

PROBLEMS FOR SECTION 2.3

1. Solve the following quadratic equations (if they have solutions):

   (a) $15x - x^2 = 0$ (b) $p^2 - 16 = 0$ (c) $(q - 3)(q + 4) = 0$

   (d) $2x^2 + 9 = 0$ (e) $x(x + 1) = 2x(x - 1)$ (f) $x^2 - 4x + 4 = 0$

2. Solve the following quadratic equations by using the method of completing the square, and factor (if possible) the left-hand side:

   (a) $x^2 - 5x + 6 = 0$ (b) $y^2 - y - 12 = 0$ (c) $2x^2 + 60x + 800 = 0$

   (d) $-\frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{2} = 0$ (e) $m(m - 5) - 3 = 0$ (f) $0.1p^2 + p - 2.4 = 0$

Solve the equations in 3–4 by using the quadratic formula:

3. (a) $r^2 + 11r - 26 = 0$ (b) $3p^2 + 45p = 48$ (c) $20\,000 = 300K - K^2$

   (d) $r^2 + (\sqrt{3} - \sqrt{2})r = \sqrt{6}$ (e) $0.3x^2 - 0.09x = 0.12$ (f) $\frac{1}{24} = p^2 - \frac{1}{12}p$

4. (a) $x^2 - 3x + 2 = 0$ (b) $5t^2 - t = 3$ (c) $6x = 4x^2 - 1$

   (d) $9x^2 + 42x + 44 = 0$ (e) $30\,000 = x(x + 200)$ (f) $3x^2 = 5x - 1$

SM 5. (a) Find the lengths of the sides of a rectangle whose perimeter is 40 cm and whose area is 75 cm^2.

   (b) Find two successive natural numbers whose sum of squares is 13.

   (c) In a right-angled triangle, the hypotenuse is 34 cm. One of the short sides is 14 cm longer than the other. Find the lengths of the two short sides.

   (d) A motorist drove 80 km. In order to save 16 minutes, he had to drive 10 km/h faster than usual. What was his usual driving speed?

6. Solve the following equations:

   (a) $x^3 - 4x = 0$ (b) $x^4 - 5x^2 + 4 = 0$ (c) $z^{-2} - 2z^{-1} - 15 = 0$