Here a, b, c, d, e, and f are arbitrary given numbers, whereas x and y are the unknowns. If we let $a = 2$, $b = 3$, $c = 18$, $d = 3$, $e = -4$, and $f = -7$, then this reduces to system (∗). Using Method 2, we multiply the first equation by e and the second by $-b$ to obtain

$$\begin{array}{rcl} aex + bey &=& ce \\ -bdx - bey &=& -bf \\ \hline (ae - bd)x &=& ce - bf \end{array}$$

which gives the value for x. We can substitute back in (1) to find y, and the result is

$$x = \frac{ce - bf}{ae - bd}, \qquad y = \frac{af - cd}{ae - bd} \tag{2}$$

We have found expressions for both x and y.

These formulas break down if the denominator $ae - bd$ in both fractions is equal to 0. This case requires special attention.

PROBLEMS FOR SECTION 2.4

Solve the systems of equations in 1–3:

1. (a) $\begin{aligned} x - y &= 5 \\ x + y &= 11 \end{aligned}$ (b) $\begin{aligned} 4x - 3y &= 1 \\ 2x + 9y &= 4 \end{aligned}$ (c) $\begin{aligned} 3x + 4y &= 2.1 \\ 5x - 6y &= 7.3 \end{aligned}$

2. (a) $\begin{aligned} 5x + 2y &= 3 \\ 2x + 3y &= -1 \end{aligned}$ (b) $\begin{aligned} x - 3y &= -25 \\ 4x + 5y &= 19 \end{aligned}$ (c) $\begin{aligned} 2x + 3y &= 3 \\ 6x + 6y &= -1 \end{aligned}$

3. (a) $\begin{aligned} 2K + L &= 11.35 \\ K + 4L &= 25.8 \end{aligned}$ (b) $\begin{aligned} 23p + 45q &= 181 \\ 10p + 15q &= 65 \end{aligned}$ (c) $\begin{aligned} 0.01r + 0.21s &= 0.042 \\ -0.25r + 0.55s &= -0.47 \end{aligned}$

SM 4. (a) Find two numbers whose sum is 52 and whose difference is 26.

(b) Five tables and 20 chairs cost \$1800, whereas 2 tables and 3 chairs cost \$420. What is the price of each table and each chair?

(c) A firm produces a good in two qualities, A and B. For the coming year, the estimated output of A is 50% higher than that of B. The profit per unit sold is \$300 for A and \$200 for B. If the profit target is \$13 000 over the next year, how much of each of the two qualities must be produced?

(d) At the beginning of the year a person had a total of \$10 000 in two accounts. The interest rates were 5% and 7.2% per year, respectively. If the person has made no transfers during the year, and has earned a total of \$676 interest, what was the initial balance in each of the two accounts?

Consider finally some equations involving fractions. Recall that the fraction a/b is not defined if $b = 0$. If $b \neq 0$, however, then $a/b = 0$ is equivalent to $a = 0$.

EXAMPLE 3 Solve the following equations:

(a) $\dfrac{1 - K^2}{\sqrt{1 + K^2}} = 0$ (b) $\dfrac{45 + 6r - 3r^2}{(r^4 + 2)^{3/2}} = 0$ (c) $\dfrac{x^2 - 5x}{\sqrt{x^2 - 25}} = 0$

Solution:

(a) The denominator is never 0, so the fraction is 0 when $1 - K^2 = 0$, that is when $K = \pm 1$.

(b) Again the denominator is never 0. The fraction is 0 when $45 + 6r - 3r^2 = 0$, that is $3r^2 - 6r - 45 = 0$. Solving this quadratic equation, we find that $r = -3$ or $r = 5$.

(c) The numerator is equal to $x(x - 5)$, which is 0 if $x = 0$ or $x = 5$. At $x = 0$ the denominator is $\sqrt{-25}$, which is not defined, and at $x = 5$ the denominator is 0. We conclude that the equation has no solutions. ▮

PROBLEMS FOR SECTION 2.5

Solve the equations in Problems 1–2:

1. (a) $x(x + 3) = 0$ (b) $x^3(1 + x^2)(1 - 2x) = 0$ (c) $x(x - 3) = x - 3$

(d) $\sqrt{2x + 5} = 0$ (e) $\dfrac{x^2 + 1}{x(x + 1)} = 0$ (f) $\dfrac{x(x + 1)}{x^2 + 1} = 0$

(SM) **2.** (a) $\dfrac{5 + x^2}{(x - 1)(x + 2)} = 0$ (b) $1 + \dfrac{2x}{x^2 + 1} = 0$

(c) $\dfrac{(x + 1)^{1/3} - \frac{1}{3}x(x + 1)^{-2/3}}{(x + 1)^{2/3}} = 0$ (d) $\dfrac{x}{x - 1} + 2x = 0$

(SM) **3.** Examine what conclusions can be drawn about the variables if:

(a) $z^2(z - a) = z^3(a + b),\ a \neq 0$ (b) $(1 + \lambda)\mu x = (1 + \lambda)y\mu$

(c) $\dfrac{\lambda}{1 + \mu} = \dfrac{-\lambda}{1 - \mu^2}$ (d) $ab - 2b - \lambda b(2 - a) = 0$

REVIEW PROBLEMS FOR CHAPTER 2

In Problems 1–2, solve each of the equations.

1. (a) $3x - 20 = 16$ (b) $-5x + 8 + 2x = -(4 - x)$ (c) $-6(x - 5) = 6(2 - 3x)$

(d) $\dfrac{4 - 2x}{3} = -5 - x$ (e) $\dfrac{5}{2x - 1} = \dfrac{1}{2 - x}$ (f) $\sqrt{x - 3} = 6$

SM 2. (a) $\dfrac{x-3}{x-4} = \dfrac{x+3}{x+4}$ (b) $\dfrac{3(x+3)}{x-3} - 2 = 9\dfrac{x}{x^2-9} + \dfrac{27}{(x+3)(x-3)}$

(c) $\dfrac{2x}{3} = \dfrac{2x-3}{3} + \dfrac{5}{x}$ (d) $\dfrac{x-5}{x+5} - 1 = \dfrac{1}{x} - \dfrac{11x+20}{x^2-5x}$

3. Solve the following equations for the variables specified:

(a) $x = \frac{2}{3}(y-3) + y$ for y (b) $ax - b = cx + d$ for x

(c) $AK\sqrt{L} = Y_0$ for L (d) $px + qy = m$ for y

(e) $\dfrac{\dfrac{1}{1+r} - a}{\dfrac{1}{1+r} + b} = c$ for r (f) $Px(Px+Q)^{-1/3} + (Px+Q)^{2/3} = 0$ for x

4. Consider the macro model

$$\text{(i) } Y = C + \bar{I} + G, \qquad \text{(ii) } C = b(Y-T), \qquad \text{(iii) } T = tY$$

where the parameters b and t lie in the interval (0, 1), Y is the gross domestic product (GDP), C is consumption, $\bar{I}$ is total investment, T denotes taxes, and G is government expenditure.

(a) Express Y and C in terms of $\bar{I}$, G, and the parameters.

(b) What happens to Y and C as t increases?

SM 5. Solve the following equations for the variables indicated:

(a) $3K^{-1/2}L^{1/3} = 1/5$ for K (b) $(1+r/100)^t = 2$ for r

(c) $p - abx_0^{b-1} = 0$ for x_0 (d) $\left[(1-\lambda)a^{-\rho} + \lambda b^{-\rho}\right]^{-1/\rho} = c$ for b

6. Solve the following quadratic equations:

(a) $z^2 = 8z$ (b) $x^2 + 2x - 35 = 0$ (c) $p^2 + 5p - 14 = 0$

(d) $12p^2 - 7p + 1 = 0$ (e) $y^2 - 15 = 8y$ (f) $42 = x^2 + x$

7. Solve the following equations:

(a) $(x^2-4)\sqrt{5-x} = 0$ (b) $(x^4+1)(4+x) = 0$ (c) $(1-\lambda)x = (1-\lambda)y$

8. Johnson invested \$1500, part of it at 15% interest and the remainder at 20%. His total yearly income from the two investments was \$275. How much did he invest at each rate?

9. If $5^{3x} = 25^{y+2}$ and $x - 2y = 8$, then what is $x - y$?

HARDER PROBLEM

10. Solve the following systems of equations:

(a) $\begin{aligned} \frac{2}{x} + \frac{3}{y} &= 4 \\ \frac{3}{x} - \frac{2}{y} &= 19 \end{aligned}$ (b) $\begin{aligned} 3\sqrt{x} + 2\sqrt{y} &= 2 \\ 2\sqrt{x} - 3\sqrt{y} &= \frac{1}{4} \end{aligned}$ (c) $\begin{aligned} x^2 + y^2 &= 13 \\ 4x^2 - 3y^2 &= 24 \end{aligned}$

Implication (a) is true (because $a = b \Longrightarrow a^2 = b^2$ and $\left(\sqrt{a}\right)^2 = a$). *It is important to note, however, that the implication cannot be replaced by an equivalence.* If $a^2 = b^2$, then either $a = b$ or $a = -b$; it need not be true that $a = b$. Implications (b), (c), (d), and (e) are also all true; moreover, all could have been written as equivalences, though this is not necessary in order to find the solution. Therefore, a chain of implications has been obtained that leads from the equation $x + 2 = \sqrt{4 - x}$ to the proposition "$x = 0$ or $x = -5$". Because the implication (a) cannot be reversed, there is no corresponding chain of implications going in the opposite direction. We have verified that if the number x satisfies $x + 2 = \sqrt{4 - x}$, then x must be either 0 or -5; no other value can satisfy the given equation. However, we have not yet shown that either 0 or -5 really satisfies the equation. Only after we try inserting 0 and -5 into the equation do we see that $x = 0$ is the only solution. *Note that in this case, the test we have suggested not only serves to check our calculations, but is also a logical necessity.*

Looking back at the wrong "solution" to Example 1, we now realize that the false argument involved two errors. Firstly, the implication $x^2 + 5x = 0 \Rightarrow x + 5 = 0$ is wrong, because $x = 0$ is also a solution of $x^2 + 5x = 0$. Secondly, it is logically necessary to check if 0 or -5 really satisfies the equation.

The method used in solving Example 4 is the most common. It involves setting up a chain of implications that starts from the given equation and ends with all the possible solutions. By testing each of these trial solutions in turn, we find which of them really do satisfy the equation. Even if the chain of implications is also a chain of equivalences, such a test is always a useful check of both logic and calculations.

PROBLEMS FOR SECTION 3.4

1. There are many other ways to express implications and equivalences, apart from those already mentioned. Use appropriate implication or equivalence arrows to represent the following propositions:

   (a) The equation $2x - 4 = 2$ is fulfilled only when $x = 3$.

   (b) If $x = 3$, then $2x - 4 = 2$.

   (c) The equation $x^2 - 2x + 1 = 0$ is satisfied if $x = 1$.

   (d) If $x^2 > 4$, then $|x| > 2$, and conversely.

2. Solve the equation $\dfrac{(x+1)^2}{x(x-1)} + \dfrac{(x-1)^2}{x(x+1)} - 2\dfrac{3x+1}{x^2-1} = 0$.

3. Consider the following six implications and decide in each case: (i) if the implication is true; and (ii) if the converse implication is true. (x, y, and z are real numbers.)

   (a) $x = 2$ and $y = 5 \Longrightarrow x + y = 7$

   (b) $(x-1)(x-2)(x-3) = 0 \Longrightarrow x = 1$

   (c) $x^2 + y^2 = 0 \Longrightarrow x = 0$ or $y = 0$

   (d) $x = 0$ and $y = 0 \Longrightarrow x^2 + y^2 = 0$

   (e) $xy = xz \Longrightarrow y = z$

   (f) $x > y^2 \Longrightarrow x > 0$

4. Consider the proposition $2x + 5 \geq 13$.

(a) Is the condition $x \geq 0$ necessary, or sufficient, or both necessary and sufficient for the inequality to be satisfied?

(b) Answer the same question when $x \geq 0$ is replaced by $x \geq 50$.

(c) Answer the same question when $x \geq 0$ is replaced by $x \geq 4$.

5. Solve the following equations:

(a) $x + 2 = \sqrt{4x + 13}$ (b) $|x + 2| = \sqrt{4 - x}$ (c) $x^2 - 2|x| - 3 = 0$

SM **6.** Solve the following equations:

(a) $\sqrt{x - 4} = \sqrt{x + 5} - 9$ (b) $\sqrt{x - 4} = 9 - \sqrt{x + 5}$

SM **7.** Fill in the blank rectangles with "iff" (if and only if) when this results in a true statement, or alternatively with "if" or "only if".

(a) $x = \sqrt{4}$ $\square$ $x = 2$ (b) $x(x + 3) < 0$ $\square$ $x > -3$

(c) $x^2 < 9$ $\square$ $x < 3$ (d) $x(x^2 + 1) = 0$ $\square$ $x = 0$

(e) $x^2 > 0$ $\square$ $x > 0$ (f) $x^4 + y^4 = 0$ $\square$ $x = 0$ or $y = 0$

8. Consider the following attempt to solve the equation $x + \sqrt{x + 4} = 2$:

"From the given equation, it follows that $\sqrt{x + 4} = 2 - x$. Squaring both sides gives $x + 4 = 4 - 4x + x^2$. After rearranging the terms, it is seen that this equation implies $x^2 - 5x = 0$. Cancelling x, we obtain $x - 5 = 0$ and this equation is satisfied when $x = 5$."

(a) Mark with arrows the implications or equivalences expressed in the text. Which ones are correct?

(b) Solve the equation correctly.

HARDER PROBLEM

SM **9.** If P is a statement, the *negation* of P is denoted by $\neg P$. If P is true, then $\neg P$ is false, and vice versa. For example, the negation of the statement $2x + 3y \leq 8$ is $2x + 3y > 8$. For each of the following 6 propositions, state the negation as simply as possible.

(a) $x \geq 0$ and $y \geq 0$.

(b) All x satisfy $x \geq a$.

(c) Neither x nor y is less than 5.

(d) For each $\varepsilon > 0$, there exists a $\delta > 0$ such that B is satisfied.

(e) No one can help liking cats.

(f) Everyone loves somebody some of the time.

PROBLEMS FOR SECTION 4.2

SM **1.** (a) Let $f(x) = x^2 + 1$. Compute $f(0)$, $f(-1)$, $f(1/2)$, and $f(\sqrt{2})$.

(b) For what values of x is it true that

(i) $f(x) = f(-x)$? (ii) $f(x+1) = f(x) + f(1)$? (iii) $f(2x) = 2f(x)$?

2. Suppose $F(x) = 10$, for all x. Find $F(0)$, $F(-3)$, and $F(a+h) - F(a)$.

3. Let $f(t) = a^2 - (t-a)^2$, where a is a constant.

(a) Compute $f(0)$, $f(a)$, $f(-a)$, and $f(2a)$. (b) Compute $3f(a) + f(-2a)$.

4. (a) For $f(x) = \dfrac{x}{1+x^2}$, compute $f(-1/10)$, $f(0)$, $f(1/\sqrt{2})$, $f(\sqrt{\pi})$, and $f(2)$.

(b) Show that $f(-x) = -f(x)$ for all x, and that $f(1/x) = f(x)$ for $x \neq 0$.

5. Let $F(t) = \sqrt{t^2 - 2t + 4}$. Compute $F(0)$, $F(-3)$, and $F(t+1)$.

6. The cost of producing x units of a commodity is given by $C(x) = 1000 + 300x + x^2$.

(a) Compute $C(0)$, $C(100)$, and $C(101) - C(100)$.

(b) Compute $C(x+1) - C(x)$, and explain in words the meaning of the difference.

7. (a) H. Schultz has estimated the demand for cotton in the US for the period 1915–1919 to be $D(P) = 6.4 - 0.3P$ (with appropriate units for the price P and the quantity $D(P)$). Find the demand in each case if the price is 8, 10, and 10.22.

(b) If the demand is 3.13, what is the price?

8. (a) If $f(x) = 100x^2$, show that for all t, $f(tx) = t^2 f(x)$.

(b) If $P(x) = x^{1/2}$, show that for all $t \geq 0$, $P(tx) = t^{1/2} P(x)$.

9. (a) The cost of removing $p\%$ of the impurities in a lake is given by $b(p) = \dfrac{10p}{105 - p}$. Find $b(0)$, $b(50)$, and $b(100)$.

(b) What does $b(50+h) - b(50)$ mean (where $h \geq 0$)?

10. Only for very special "additive" functions is it true that $f(a+b) = f(a) + f(b)$ for all a and b. Determine whether $f(2+1) = f(2) + f(1)$ for the following:

(a) $f(x) = 2x^2$ (b) $f(x) = -3x$ (c) $f(x) = \sqrt{x}$

11. (a) If $f(x) = Ax$, show that $f(a+b) = f(a) + f(b)$ for all a and b.

(b) If $f(x) = 10^x$, show that $f(a+b) = f(a) \cdot f(b)$ for all natural numbers a and b.

12. A student claims that $(x+1)^2 = x^2 + 1$. Can you use a geometric argument to show that this is wrong?

(SM) 13. Find the domains of the functions defined by the following formulas:

(a) $y = \sqrt{5-x}$ (b) $y = \dfrac{2x-1}{x^2-x}$ (c) $y = \sqrt{\dfrac{x-1}{(x-2)(x+3)}}$

14. (a) Find the domain of the function f defined by the formula $f(x) = \dfrac{3x+6}{x-2}$.

(b) Show that the number 5 is in the range of f by finding a number x such that $(3x+6)/(x-2) = 5$.

(c) Show that the number 3 is not in the range of f.

15. Find the domain and the range of $g(x) = 1 - \sqrt{x+2}$.

4.3 Graphs of Functions

Recall that a **rectangular** (or a **Cartesian**) coordinate system is obtained by first drawing two perpendicular lines, called coordinate axes. The two axes are respectively the *x-axis* (or the *horizontal axis*) and the *y-axis* (or the *vertical axis*). The intersection point O is called the *origin*. We measure the real numbers along each of these lines, as shown in Fig. 1. The unit distance on the x-axis is not necessarily the same as on the y-axis, although this is the case in Fig. 1.

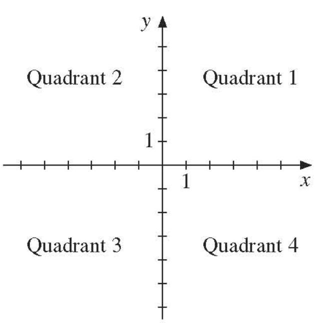

Figure 1 A coordinate system

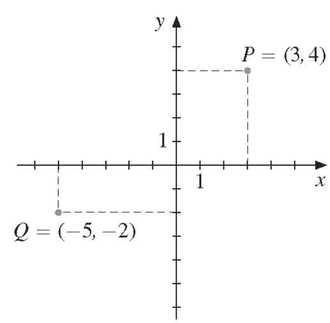

Figure 2 Points $(3, 4)$ and $(-5, -2)$

The rectangular coordinate system in Fig. 1 is also called the ***xy*-plane**. The coordinate axes separate the plane into four quadrants, which traditionally are numbered as in Fig. 1. Any point P in the plane can be represented by a unique pair (a, b) of real numbers. These can be found by drawing dashed lines, like those in Figure 2, which are perpendicular to the two axes. The point represented by (a, b) lies at the intersection of the vertical straight line $x = a$ with the horizontal straight line $y = b$.

EXAMPLE 7 A person has $\$m$ to spend on the purchase of two commodities. The prices of the two commodities are $\$p$ and $\$q$ per unit. Suppose x units of the first commodity and y units of the second commodity are bought. Assuming that negative purchases of either commodity are impossible, one must have both $x \geq 0$ and $y \geq 0$. It follows that the person is restricted to the *budget set* given by

$$B = \{(x, y) : \ px + qy \leq m, \ x \geq 0, \ y \geq 0\}$$

as in (3.6.1). Sketch the budget set B in the xy-plane. Find the slope of the budget line $px + qy = m$, and its x- and y-intercepts.

Solution: The set of points (x, y) that satisfy $x \geq 0$ and $y \geq 0$ is the first (nonnegative) quadrant. If we impose the additional requirement that $px+qy \leq m$, we obtain the triangular domain B shown in Fig. 12.

If $px + qy = m$, then $qy = -px + m$ and so $y = (-p/q)x + m/q$. This shows that the slope is $-p/q$. The budget line intersects the x-axis when $y = 0$. Then $px = m$, so $x = m/p$. The budget line intersects the y-axis when $x = 0$. Then $qy = m$, so $y = m/q$. So the two points of intersection are $(m/p, 0)$ and $(0, m/q)$, as shown in Fig. 12. ▮

PROBLEMS FOR SECTION 4.4

1. Find the slopes of the lines passing through the following pairs of points:

(a) $(2, 3)$ and $(5, 8)$ (b) $(-1, -3)$ and $(2, -5)$ (c) $\left(\frac{1}{2}, \frac{3}{2}\right)$ and $\left(\frac{1}{3}, -\frac{1}{5}\right)$

2. Draw graphs for the following straight lines:

(a) $3x + 4y = 12$ (b) $\dfrac{x}{10} - \dfrac{y}{5} = 1$ (c) $x = 3$

3. Suppose demand D for a good is a linear function of its price per unit, P. When price is \$10, demand is 200 units, and when price is \$15, demand is 150 units. Find the demand function.

4. Find the slopes of the five lines L_1 to L_5 shown in the figure, and give equations describing them.

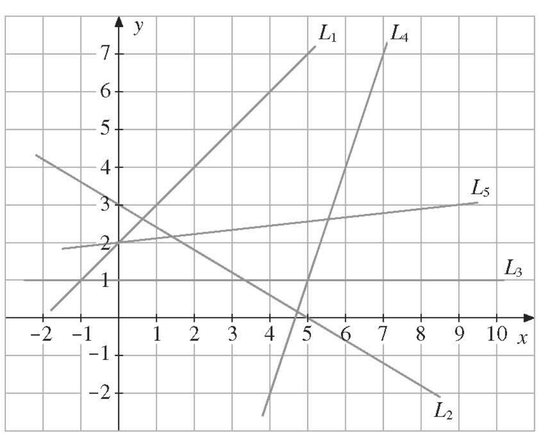

Figure 13

5. Decide which of the following relationships are linear:

(a) $5y + 2x = 2$ (b) $P = 10(1 - 0.3t)$ (c) $C = (0.5x + 2)(x - 3)$

(d) $p_1x_1 + p_2x_2 = R$ (p_1, p_2, and R constants)

6. A printing company quotes the price of \$1400 for producing 100 copies of a report, and \$3000 for 500 copies. Assuming a linear relation, what would be the price of printing 300 copies?

7. Determine the equations for the following straight lines:

(a) L_1 passes through $(1, 3)$ and has a slope of 2.

(b) L_2 passes through $(-2, 2)$ and $(3, 3)$.

(c) L_3 passes through the origin and has a slope of $-1/2$.

(d) L_4 passes through $(a, 0)$ and $(0, b)$ (suppose $a \neq 0$).

8. Sketch in the xy-plane the set of all pairs of numbers (x, y) that satisfy the following inequalities:

(a) $2x + 4y \geq 5$ (b) $x - 3y + 2 \leq 0$ (c) $100x + 200y \leq 300$

9. Solve the following three systems of equations graphically:

(a) $\begin{aligned} x - y &= 5 \\ x + y &= 1 \end{aligned}$ (b) $\begin{aligned} x + y &= 2 \\ x - 2y &= 2 \\ x - y &= 2 \end{aligned}$ (c) $\begin{aligned} 3x + 4y &= 1 \\ 6x + 8y &= 6 \end{aligned}$

SM 10. Sketch in the xy-plane the set of all pairs of numbers (x, y) that satisfy all the following three inequalities:

$$3x + 4y \leq 12, \quad x - y \leq 1, \quad \text{and} \quad 3x + y \geq 3$$

4.5 Linear Models

Linear relations occur frequently in mathematical models. The relationship between the Celsius and Fahrenheit temperature scales, $F = \frac{9}{5}C + 32$ (see Example 1.6.4), is an example of an exact (by definition) linear relation between two variables. Most of the linear models in economics are approximations to more complicated models. Two typical relations are those shown in Example 4.4.1. Statistical methods have been devised to construct linear functions that approximate the actual data as closely as possible. Let us consider a very naive attempt to construct a linear model based on some population data.

EXAMPLE 1 A United Nations report estimated that the European population was 641 million in 1960, and 705 million in 1970. Use these estimates to construct a linear function of t that approximates the population in Europe (in millions), where t is the number of years from 1960 ($t = 0$ is 1960, $t = 1$ is 1961, and so on). Then use the function to estimate the population in 1975, 2000, and 1930.

Here a and b are positive parameters of the demand function D, while α and β are positive parameters of the supply function.

Such linear supply and demand functions play an important role in economics. It is often the case that the market for a particular commodity, such as copper, can be represented approximately by suitably estimated linear demand and supply functions.

The equilibrium price P^e occurs where demand equals supply. Hence $D = S$ at $P = P^e$ implying that $a - bP^e = \alpha + \beta P^e$, or $a - \alpha = (\beta + b)P^e$. The corresponding equilibrium quantity is $Q^e = a - bP^e$. So equilibrium occurs at

$$P^e = \frac{a-\alpha}{\beta+b}, \qquad Q^e = a - b\frac{a-\alpha}{\beta+b} = \frac{a\beta+\alpha b}{\beta+b}$$

PROBLEMS FOR SECTION 4.5

1. The consumption function $C = 4141 + 0.78Y$ was estimated for the UK during the period 1949–1975. What is the marginal propensity to consume?

2. Find the equilibrium price for each of the two linear models of supply and demand:

   (a) $D = 75 - 3P, \quad S = 20 + 2P$ (b) $D = 100 - 0.5P, \quad S = 10 + 0.5P$

3. The total cost C of producing x units of some commodity is a linear function of x. Records show that on one occasion, 100 units were made at a total cost of \$200, and on another occasion, 150 units were made at a total cost of \$275. Express the linear equation for total cost C in terms of the number of units x produced.

4. The expenditure of a household on consumer goods, C, is related to the household's income, y, in the following way: When the household's income is \$1000, the expenditure on consumer goods is \$900, and whenever income increases by \$100, the expenditure on consumer goods increases by \$80. Express the expenditure on consumer goods as a function of income, assuming a linear relationship.

5. For most assets such as cars, stereo equipment, and furniture, the value decreases, or *depreciates*, each year. If the value of an asset is assumed to decrease by a fixed percentage of the original value each year, it is referred to as *straight line depreciation*.

   (a) Suppose the value of a car which initially costs \$20 000 depreciates by 10% of its original value each year. Find a formula for its value $P(t)$ after t years.

   (b) If a \$500 washing machine is completely depreciated after 10 years (straight line depreciation), find a formula for its value $W(t)$ after t years.

6. (a) According to the 20th report of the International Commission on Whaling, the number N of fin whales in the Antarctic for the period 1958–1963 was estimated to be

   $$N = -17\,400t + 151\,000, \qquad 0 \le t \le 5$$

   where $t = 0$ corresponds to January 1958, $t = 1$ corresponds to January 1959, and so on. According to this equation, how many fin whales would be left in April 1960?

   (b) If the decrease continued at the same rate, when would there be no fin whales left? (Actually, the 1993 estimate was approximately 21 000.)

subsidizing the output of the monopolist in order to encourage additional production. (Of course, subsidizing monopolists is usually felt to be unjust, and many additional complications need to be considered carefully before formulating a desirable policy for dealing with monopolists. Still the previous analysis suggests that if it is desirable to lower a monopolist's price or its profit, this is much better done directly than by taxing its output.) ∎

PROBLEMS FOR SECTION 4.6

1. (a) Let $f(x) = x^2 - 4x$. Complete the following table and use it to sketch the graph of f:

x	-1	0	1	2	3	4	5
$f(x)$							

(b) Using (3), determine the minimum point of f.

(c) Solve the equation $f(x) = 0$.

2. (a) Let $f(x) = -\frac{1}{2}x^2 - x + \frac{3}{2}$. Complete the following table and sketch the graph of f:

x	-4	-3	-2	-1	0	1	2
$f(x)$							

(b) Using (4), determine the maximum point of f.

(c) Solve the equation $-\frac{1}{2}x^2 - x + \frac{3}{2} = 0$ for x.

(d) Show that $f(x) = -\frac{1}{2}(x - 1)(x + 3)$, and use this to study how the sign of $f(x)$ varies with x. Compare the result with the graph.

3. Determine the maximum/minimum points by using (3) or (4):

(a) $x^2 + 4x$ (b) $x^2 + 6x + 18$ (c) $-3x^2 + 30x - 30$

(d) $9x^2 - 6x - 44$ (e) $-x^2 - 200x + 30\,000$ (f) $x^2 + 100x - 20\,000$

4. Find all the zeros of each quadratic function in Problem 3, and write each function in the form $a(x - x_1)(x - x_2)$ (if possible).

5. Find solutions to the following equations, where p and q are parameters.

(a) $x^2 - 3px + 2p^2 = 0$ (b) $x^2 - (p + q)x + pq = 0$ (c) $2x^2 + (4q - p)x = 2pq$

6. A model by A. Sandmo in the theory of efficient loan markets involves the function

$$U(x) = 72 - (4 + x)^2 - (4 - rx)^2$$

where r is a constant. Find the value of x for which $U(x)$ attains its largest value.

7. (a) A farmer has 1000 metres of fence wire with which to make a rectangular enclosure, as illustrated in the figure below. Find the areas of the three rectangles whose bases are 100, 250, and 350 metres.

(b) Let the base have length $250 + x$. Then the height is $250 - x$ (see Fig. 5). What choice of x gives the maximum area?[4]

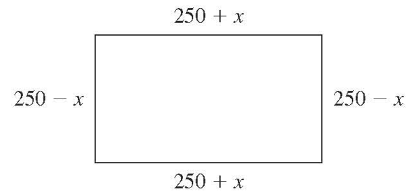

Figure 5

8. (a) If a cocoa shipping firm sells Q tons of cocoa in the UK, the price received is given by $P_E = \alpha_1 - \frac{1}{3}Q$. On the other hand, if it buys Q tons from its only source in Ghana, the price it has to pay is given by $P_G = \alpha_2 + \frac{1}{6}Q$. In addition, it costs γ per ton to ship cocoa from its supplier in Ghana to its customers in the UK (its only market). The numbers α_1, α_2, and γ are all positive. Express the cocoa shipper's profit as a function of Q, the number of tons shipped.

(b) Assuming that $\alpha_1 - \alpha_2 - \gamma > 0$, find the profit-maximizing shipment of cocoa. What happens if $\alpha_1 - \alpha_2 - \gamma \leq 0$?

(c) Suppose the government of Ghana imposes an export tax on cocoa of t per ton. Find the new expression for the shipper's profits and the new quantity shipped.

(d) Calculate the Ghanaian government's export tax revenue T as a function of t, and compare the graph of this function with the Laffer curve presented in Section 4.1.

(e) Advise the Ghanaian government on how to obtain as much tax revenue as possible.

HARDER PROBLEM

(SM) 9. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be arbitrary real numbers. We claim that the following inequality (called the **Cauchy–Schwarz inequality**) is always valid:

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \tag{5}$$

(a) Check the inequality for $a_1 = -3$, $a_2 = 2$, $b_1 = 5$, and $b_2 = -2$. (Then $n = 2$.)

(b) Prove (5) by means of the following trick: first, define f for all x by

$$f(x) = (a_1x + b_1)^2 + \cdots + (a_nx + b_n)^2$$

It should be obvious that $f(x) \geq 0$ for all x. Write $f(x)$ as $Ax^2 + Bx + C$, where the expressions for A, B, and C are related to the terms in (5). Because $Ax^2 + Bx + C \geq 0$ for all x, we must have $B^2 - 4AC \leq 0$. Why? The conclusion follows.

[4] It is reported that certain surveyors in antiquity wrote contracts with farmers to sell them rectangular pieces of land in which only the perimeter was specified. As a result, the lots were long narrow rectangles.