

Constrained Optimization: The Method of Lagrange Multipliers:

Suppose the equation $p(x,y) = -2x^2 + 60x - 3y^2 + 72y + 100$ models profit when x represents the number of handmade chairs and y is the number of handmade rockers produced per week. The optimal (maximum) situation occurs when $x = 15$ and $y = 12$. However due to an insufficient labor force they can only make a total of 20 chairs and rockers per week ($x + y = 20$). So how many chairs and how many rockers will give the realistic maximum profit? We will come back to this questions shortly but first we will look at the following example.

Using Level Curves and the constraint function to determine optimal points:

A Company has a determined that its production function is the Cobb-Douglas function $f(x,y) = x^{\frac{2}{3}}y^{\frac{1}{3}}$ where x is the number of labor hours and y is the number of capitol units. The budget constraint for the company is given by $100x + 100y = 400000$.

a) If the company decides to spend \$300, 000 on x then how much can be spent on y under this budget constraint?

solution:

If \$300000 is spent on x then $100x = 300000$ and $x = 3000$. There is \$100000 left to spend on y therefore $100y = 100000$ and $y = 1000$.

In this case total production will be $f(3000, 1000) = (3000)^{\frac{2}{3}}(1000)^{\frac{1}{3}} = (208)(10) = 2080$ units.

Suppose the amount spent on x is changed to \$350,000. How will that change production? Following the same procedure as above $x = 3500$ and $y = 500$. The production $f(3500, 500)$ then be 1829 units.

Is there a way to determine values for x and y that will give the optimal production possible given this budget constraint?

we will first try to find the optimal situation using level curves with the constraint function.

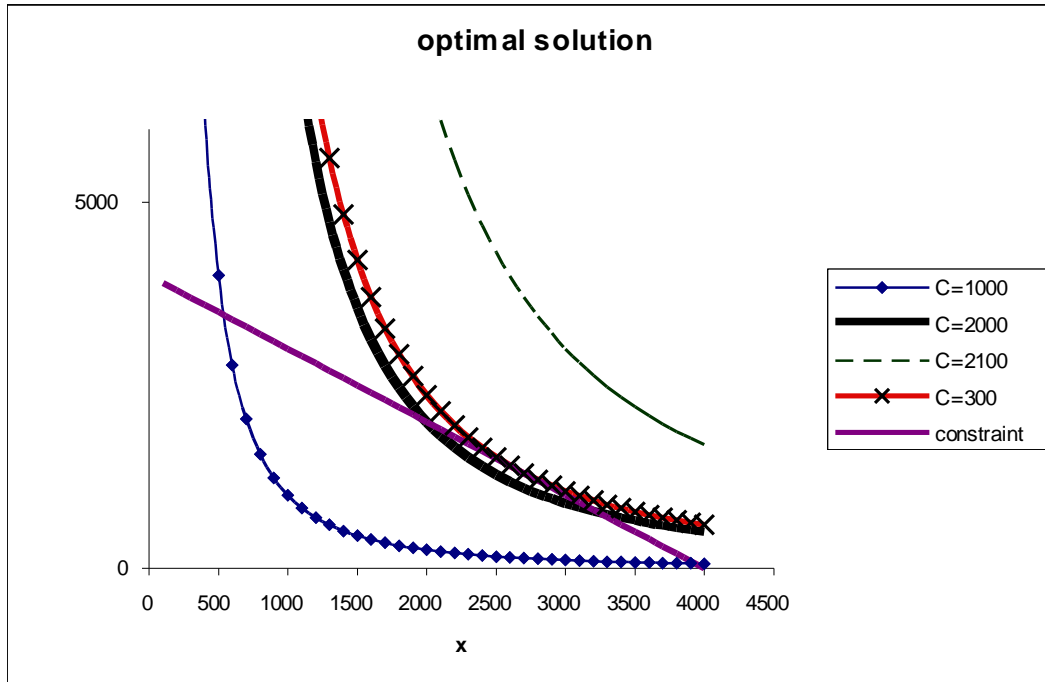
It turns out that the global maximum or global minimum occurs where the graph of the constraint equation is tangent to one of the level curves of the original function.

In order to plot the level curves we must solve for y to get $\frac{f(x,y)}{x^{\frac{2}{3}}} = y^{\frac{1}{3}} \rightarrow \frac{C^3}{x^2} = y$

where C now represents values of $f(x,y)$. Now we will plot y for various values of C . (The choice of values for C is determined mostly by common sense.) In this case we

will let $C = 1000$, $C = 2000$ and $C = 3000$. After looking at the resulting level curves and constraint, I decide to add another value $C = 2100$. Now I can make a good guess

After considering the graph below, we can guess the optimal value will occur for $x = 2600$ and $y = 1400$.



There is also an algebraic approach to finding the optimal solution given a certain constraint. We will use the method of Lagrange Multipliers to find the maximum situation in the problem above. In the proceeding sections you have learned how to use partial derivatives to find the optimal situation for a multivariable equation. Now we will expand on this process to find the optimal situation with a constraint. Below is an outline of the process we will use followed examples.

Procedure for Applying the Method of Lagrange Multipliers:

In order to maximize or minimize the function $f(x,y)$ which is subject to the constraint $g(x, y) = k$ we will follow the following procedure.

Step #1 First create the LaGrange Function. This function is composed of the function to be optimized combined with the constraint function in the following way:

$$L(x, y) = f(x, y) - \lambda[g(x, y) - k]$$

Step #2 Now find the partial derivative with respect to each variable x , y and the Lagrange multiplier λ of the function shown: $L(x, y) = f(x, y) - \lambda[g(x, y) - k]$

Step #3 Set each of the partial derivatives equal to zero to get $L_x = 0$, $L_y = 0$ and $L_\lambda = 0$. Using $L_x = 0$, $L_y = 0$, proceed to solve for x and solve for y in terms of λ . Now substitute the solutions for x and y so that $L_\lambda = 0$ is in terms of λ only. Now solve for λ and use this value to find the optimal values x and y .

Note: If M is the max or min value of $f(x, y)$ subject to the constraint $g(x, y) = k$, then the Lagrange multiplier λ is the rate of change in M with respect to k . (i.e. $\lambda = dM/dk$ and therefore λ approximates the change in M resulting in a one unit increase in k .)

Example 1

We will revisit the Cobb-Douglas function $f(x,y) = x^{\frac{2}{3}}y^{\frac{1}{3}}$ and the budget constraint function $100x + 100y = 400000$. We will use the method of Lagrange multiplier described above to find the optimal solutions.

First we will create the Lagrange equation

$$L(x,y) = x^{\frac{2}{3}}y^{\frac{1}{3}} - \lambda(100x + 100y - 400000)$$

Now find $L_x = 0$, $L_y = 0$ and $L_\lambda = 0$.

$$L_x = \frac{2}{3} x^{-\frac{1}{3}} y^{\frac{1}{3}} - 100\lambda$$

$$L_y = \frac{1}{3} x^{\frac{2}{3}} y^{-\frac{2}{3}} - 100\lambda$$

$$L_\lambda = -100x - 100y + 400000$$

Using $L_x = 0$, $L_y = 0$ to solve for x and y

Set the first derivative $L_x = 0$ to get and solve for $\lambda = \frac{\frac{2}{3} x^{-\frac{1}{3}} y^{\frac{1}{3}}}{100}$ and then set $L_y = 0$ and

solve for λ again to get $\lambda = \frac{\frac{1}{3} x^{\frac{2}{3}} y^{-\frac{2}{3}}}{100}$

We now need to use the results to turn the equation $L_\lambda = 0$ into an equation of one variable only. To do this we will set the λ equations equal to each other and solve for y as shown below.

$$\frac{\frac{2}{3}x^{-\frac{1}{3}}y^{\frac{1}{3}}}{100} = \frac{\frac{1}{3}x^{\frac{2}{3}}y^{-\frac{2}{3}}}{100} \rightarrow \frac{\frac{2}{3}x^{-\frac{1}{3}}y^{\frac{1}{3}}}{100} = \frac{\frac{1}{3}x^{\frac{2}{3}}y^{-\frac{2}{3}}}{100} \rightarrow \text{multiply both sides by 100 and then}$$

$$\text{by 3 to get } 2x^{-\frac{1}{3}}y^{\frac{1}{3}} = x^{\frac{2}{3}}y^{-\frac{2}{3}} \rightarrow \frac{2y^{\frac{1}{3}}}{y^{\frac{-2}{3}}} = \frac{x^{\frac{2}{3}}}{x^{\frac{-1}{3}}} \rightarrow 2y = x \rightarrow y = \frac{1}{2}x$$

Now substitute the values of y into $L_\lambda = 0$ to get $-100x - 100(\frac{1}{2}x) + 400000 = 0$ and solve for $x = 2666.66$ and then find y to be $y = 1333.33$.

The value of x and y that will maximize production is $x = 2666$ and $y = 1333$ and the maximum production is 2116 units.

Example 2:

We will now revisit the profit equation introduced at the beginning of this section.

$$p(x,y) = -2x^2 + 60x - 3y^2 + 72y + 100$$

where x represents the number of handmade chairs and y is the number of handmade rockers produced per week. The constraint on this profit is $g(x, y) = x + y = 20$.

In order to find the optimal situation given this constraint we will follow the method given above. First we will create the LaGrange equation

$$L(x, y) = p(x, y) - \lambda[g(x, y) - k] = (-2x^2 + 60x - 3y^2 + 72y + 100) - \lambda(x + y - 20)$$

Now find $L_x = 0$, $L_y = 0$ and $L_\lambda = 0$.

$$L_x = -4x + 60 - \lambda = 0$$

$$L_y = -6y + 72 - \lambda = 0$$

$$L_\lambda = -x - y + 20 = 0$$

Using $L_x = 0$, $L_y = 0$ to solve for x and y to get $x = 15 - \frac{1}{4}\lambda$ and $y = 12 - \frac{1}{6}\lambda$. Now

substitute the values of x and y into $L_\lambda = 0$ to get

$$L_\lambda = -x - y + 20 = -(15 - \frac{1}{4}\lambda) - (12 - \frac{1}{6}\lambda) + 20 = (-27 + \frac{10}{24}\lambda) + 20 = -7 + \frac{5}{12}\lambda = 0 \text{ and}$$

$\lambda = 16.8$ so $x = 10.8 \approx 10$ and $y = 9.2 \approx 9$. Therefore the maximum profit will be achieved when 10 chairs and 9 rockers are produced. (Remember that the maximum profit is $x = 15$ and $y = 12$ when there is not constraint.)

Notice that the value of $\lambda = 16.8$ and if the constraint was increased from 20 to 21 then the maximum profit would increase by \$16.80

Example 3:

Amanda is getting a new dog. She wants to build a pen for her dog in the back yard. The pen will be rectangular using 200 feet of fence. Amanda plans to build the pen up against the wall of her house so that she will only need three sides of fence. She wants to build a pen with the maximum amount of area. Therefore we need to maximize the area equation $A = xy$.

Since Amanda has a constraint in the amount of fencing she can use, we will use the LaGrange method and create the following equation.

$$L(x,y) = xy - \lambda(x + y - 200)$$

Now find $L_x = 0$, $L_y = 0$ and solve for x and y to get that $x = \lambda$ and $y = 2\lambda$. Find $L_\lambda = 0$ and substitute in the values for x and y to put the equation in terms of λ . Now solve to get that $\lambda = 50$. Therefore the optimal values are $x = 50$ and $y = 100$ feet of fencing. (Notice that if the constant 200 feet of fencing is increased by one foot to 201 feet then by marginal analysis the maximum value for the area will increase by 50 square feet. Determine what the maximum area would be for 201 feet of fencing) Answer is _____

Example 4 The function $U = f(x,y)$ represent the utility or customer satisfaction derived by a consumer from the consumption of a certain amount of product x and a certain amount of product y . In the example below the maximum utility is determined when the given budget constraint affects the amount of each product produced.

The function $f(x,y) = x^2y^2$ is a utility function with a budget constraint given to be $g(x,y) = 2x + 4y = 40$.

The first step is to create the LaGrange equation $L(x,y) = x^2y^2 - \lambda(2x + 4y - 40)$

$$L_x = \frac{\partial L}{\partial x} = 2xy^2 - 2\lambda = 0 \quad \text{and therefore } \lambda = xy^2$$

$$L_y = \frac{\partial L}{\partial y} = 2x^2y - 4\lambda = 0 \quad \text{and therefore } \lambda = \frac{1}{2}x^2y$$

The next step is to find $L_\lambda = -2x - 4y + 40 = 0$ and then use the partials above to help transform L_λ into an equation with just one variable. It does not matter if it is terms of x only, y only or λ only (as in the two examples above)

In this case it is easier to set the λ equations equal to each other and solve for either x or y as seen below.

$$\text{set } xy^2 = \frac{1}{2}x^2y \rightarrow \frac{y^2}{y} = \frac{1}{2} \frac{x^2}{x} \rightarrow y = \frac{x}{2}$$

Going back and substitute into $L_\lambda = -2x - 4y + 40 = -2x - 4\left(\frac{x}{2}\right) + 40 = 0$ and solve to get $x = 10$ and then $y = 5$. (You will also get $\lambda = xy^2 = (10)(25) = 250$. What is the meaning of this information ? It means that if the constant is increased to 41 the maximum utility will increase by 250)

Practice Problems

Problem 1

1) Find the minimum value of the function $f(x, y) = x^2 + 2y^2 - xy$ subject to the constraint $2x + y = 22$.

Problem 2

The highway Department is planning to build a picnic area for motorist along a major highway. It is rectangular with area of 5,000 square yards. It is to be fenced off on the three sides not adjacent to the highway. What is the least amount of fencing needed to complete the job?

Problem 3

Using the information given in problem 2 graph the level curves and the constraint function and indicate where the minimum point will occur.

Problem 4

Two women have decided to start a cottage industry that will be run out of their homes. They are going to make two types of specialty shirts, rhinestone and embroidery. The total gross profit is given by $P(x, y) = 16x + 20y - x^2 - y^2$ where x is the number of rhinestone shirts made and y is the number of embroidery shirts made per month. Together, they are able to make at most a total of six shirts per month. What is the number of each type of shirt that they should try to make and sell in order to maximize their profit each month.?

Problem 5

If the equation $f(x, y, z) = x^2 + y^2 + z^2$ is subject to the constraint $x - y + 2z = 6$, find the optimal point under these conditions.

Problem 5

A) An editor has been allotted \$60000 to spend on the development and promotion of a new book. It is estimated that if x thousand dollars is spent on development and y thousand on promotion, approximately $f(x, y) = 20x^2 y^{\frac{3}{5}}$ copies of the book will be sold. How much money should the editor allocate to development and how much to promotion in order to maximize sales?

C) Graph the level curves and the constraint function and determine the maximum point.

B) Suppose the editor's allocation is increased to \$61000 to spend on the development and promotion of the new book. Estimate how the additional \$1000 will affect the maximum sales level.

Problem 6

A farmer wishes to fence off a rectangular pasture along the bank of a river. The area of the pasture is to be 3,200 square meters and no fencing is needed along the bank. Find the dimensions of the pasture that will require the least amount of fencing.

Problem 7

A Cobb-Douglas production function for new company is given by $f(x, y) = 50x^{\frac{2}{5}}y^{\frac{3}{5}}$ where x represents the units of labor and y represents the units of capital. Suppose units of Labor and capital cost \$100 and \$200 each respectively. If the budget constraint is \$30,000, find the maximum production level for this manufacture.