Ex1 Initially, three firms A, B, and C (numbered 1, 2, and 3) share the market for a certain commodity. Firm A has 20% of the market, B has 60%, and C has 20%. In the course of the next year, the following changes occur:

$$\begin{cases} A \text{ keeps 85\% of its customers, while losing 5\% to } B \text{ and 10\% to } C \\ B \text{ keeps 55\% of its customers, while losing 10\% to } A \text{ and 35\% to } C \\ C \text{ keeps 85\% of its customers, while losing 10\% to } A \text{ and 5\% to } B \end{cases}$$

We can represent market shares of the three firms by means of a *market share vector*, defined as a column vector $\mathbf{s}$ whose components are all nonnegative and sum to 1. Define the matrix $\mathbf{T}$ and the initial market share vector $\mathbf{s}$ by

$$\mathbf{T} = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix} \quad \text{and} \quad \mathbf{s} = \begin{pmatrix} 0.2 \\ 0.6 \\ 0.2 \end{pmatrix}$$

Notice that t_{ij} is the percentage of j's customers who become i's customers in the next period. So $\mathbf{T}$ is called the *transition matrix*.

Compute the vector $\mathbf{Ts}$, show that it is also a market share vector, and give an interpretation. What is the interpretation of $\mathbf{T}(\mathbf{Ts})$, $\mathbf{T}(\mathbf{T}(\mathbf{Ts}))$, ...?

Exersices

15-4

1. Compute the products $\mathbf{AB}$ and $\mathbf{BA}$, if possible, for the following:

(a) $\mathbf{A} = \begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 & 4 \\ 1 & 5 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 8 & 3 & -2 \\ 1 & 0 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & -2 \\ 4 & 3 \\ 1 & -5 \end{pmatrix}$

(c) $\mathbf{A} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$, $\mathbf{B} = (0, \quad -2, \quad 3)$ (d) $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 2 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \\ 0 & 2 \end{pmatrix}$

2. Given the matrices $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$,

calculate (i) $3\mathbf{A} + 2\mathbf{B} - 2\mathbf{C} + \mathbf{D}$ (ii) $\mathbf{AB}$ (iii) $\mathbf{C}(\mathbf{AB})$.

3. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{pmatrix}$.

Find the matrices $\mathbf{A} + \mathbf{B}$, $\mathbf{A} - \mathbf{B}$, $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{A}(\mathbf{BC})$, and $(\mathbf{AB})\mathbf{C}$.

4. Write out three matrix equations corresponding to the following systems:

(a)
$$\begin{aligned} x_1 + x_2 &= 3 \\ 3x_1 + 5x_2 &= 5 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_1 - x_2 + x_3 &= 5 \\ 2x_1 + 3x_2 - x_3 &= 1 \end{aligned}$$

(c)
$$\begin{aligned} 2x_1 - 3x_2 + x_3 &= 0 \\ x_1 + x_2 - x_3 &= 0 \end{aligned}$$

5. Consider the three matrices $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 1 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}$, and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) Find a matrix $\mathbf{C}$ satisfying $(\mathbf{A} - 2\mathbf{I})\mathbf{C} = \mathbf{I}$.

(b) Is there a matrix $\mathbf{D}$ satisfying $(\mathbf{B} - 2\mathbf{I})\mathbf{D} = \mathbf{I}$?

15-5

1. Find the transposes of $\mathbf{A} = \begin{pmatrix} 3 & 5 & 8 & 3 \\ -1 & 2 & 6 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{C} = (1, 5, 0, -1)$.

2. Let $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$, and $\alpha = -2$. Compute $\mathbf{A}'$, $\mathbf{B}'$, $(\mathbf{A}+\mathbf{B})'$, $(\alpha\mathbf{A})'$, $\mathbf{AB}$, $(\mathbf{AB})'$, $\mathbf{B}'\mathbf{A}'$, and $\mathbf{A}'\mathbf{B}'$. Then verify all the rules in (2) for these particular values of $\mathbf{A}$, $\mathbf{B}$, and α.

3. Show that $\mathbf{A} = \begin{pmatrix} 3 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 4 & 8 \\ 4 & 0 & 13 \\ 8 & 13 & 0 \end{pmatrix}$ are symmetric.

4. For what values of a is $\begin{pmatrix} a & a^2-1 & -3 \\ a+1 & 2 & a^2+4 \\ -3 & 4a & -1 \end{pmatrix}$ symmetric?

5. Is the product of two symmetric matrices necessarily symmetric?

SM 6. If $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$ are matrices for which the given products are defined, show that

$$(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3)' = \mathbf{A}_3'\mathbf{A}_2'\mathbf{A}_1'$$

Generalize to products of n matrices.

7. An $n \times n$ matrix $\mathbf{P}$ is said to be **orthogonal** if $\mathbf{P}'\mathbf{P} = \mathbf{I}_n$.

   (a) For $\lambda = \pm 1/\sqrt{2}$, show that $\mathbf{P} = \begin{pmatrix} \lambda & 0 & \lambda \\ \lambda & 0 & -\lambda \\ 0 & 1 & 0 \end{pmatrix}$ is orthogonal.

   (b) Show that the 2×2 matrix $\begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ is orthogonal if and only if $p^2 + q^2 = 1$.

   (c) Show that the product of two orthogonal $n \times n$ matrices is orthogonal.

SM 8. Define the two matrices $\mathbf{T}$ and $\mathbf{S}$ by $\mathbf{T} = \begin{pmatrix} p & q & 0 \\ \frac{1}{2}p & \frac{1}{2} & \frac{1}{2}q \\ 0 & p & q \end{pmatrix}$, $\mathbf{S} = \begin{pmatrix} p^2 & 2pq & q^2 \\ p^2 & 2pq & q^2 \\ p^2 & 2pq & q^2 \end{pmatrix}$,

and assume that $p + q = 1$.

   (a) Prove that $\mathbf{T} \cdot \mathbf{S} = \mathbf{S}$, $\mathbf{T}^2 = \frac{1}{2}\mathbf{T} + \frac{1}{2}\mathbf{S}$, and $\mathbf{T}^3 = \frac{1}{4}\mathbf{T} + \frac{3}{4}\mathbf{S}$.

   (b) Conjecture formulas for constants α_n, β_n such that $\mathbf{T}^n = \alpha_n\mathbf{T} + \beta_n\mathbf{S}$ for $n = 2, 3, \ldots$, then prove the formulas by induction.

15-6

1. Solve the following systems by Gaussian elimination.

(a) $$\begin{aligned} x_1 + x_2 &= 3 \\ 3x_1 + 5x_2 &= 5 \end{aligned}$$

(b) $$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_1 - x_2 + x_3 &= 5 \\ 2x_1 + 3x_2 - x_3 &= 1 \end{aligned}$$

(c) $$\begin{aligned} 2x_1 - 3x_2 + x_3 &= 0 \\ x_1 + x_2 - x_3 &= 0 \end{aligned}$$

2. Use Gaussian elimination to discuss what are the possible solutions of the following system for different values of a and b:

$$\begin{aligned} x + y - z &= 1 \\ x - y + 2z &= 2 \\ x + 2y + az &= b \end{aligned}$$

SM **3.** Find the values of c for which the system

$$\begin{aligned} 2w + x + 4y + 3z &= 1 \\ w + 3x + 2y - z &= 3c \\ w + x + 2y + z &= c^2 \end{aligned}$$

has a solution, and find the complete solution for these values of c.

SM **4.** Consider the two systems of equations:

(a) $$\begin{aligned} ax + y + (a+1)z &= b_1 \\ x + 2y + z &= b_2 \\ 3x + 4y + 7z &= b_3 \end{aligned}$$

(b) $$\begin{aligned} \tfrac{3}{4}x + y + \tfrac{7}{4}z &= b_1 \\ x + 2y + z &= b_2 \\ 3x + 4y + 7z &= b_3 \end{aligned}$$

Find the values of a for which (a) has a unique solution, and find all solutions to system (b).